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Entropy, Superposition and Dynamical Maps

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Abstract Aspects of quantum entropy and relative quantum entropy are discussed in the Hilbert model. It is shown that finite values of the relative entropy of states implies a superposition relation between the states. The property is studied in case of tensor product of states and for state reductions. A "Schmidt-like" state, derived from the reduced states, is considered. It is shown that its entropy, relative to the product of the reduced states, is not smaller than the entropy of the reduced states. The main existing results concerning the changement of superposition and entropy under dynamical map are recalled in a uniform way. A class of possible dynamical maps, not necessarily linear, is proposed that do not decrease the entropy.

Keywords Entropy · Superposition · Reduced state · Dynamical maps · Nonlinear dynamical maps

1 Introduction

The quantum entropy and the quantum relative entropy functions has been widely studied in the literature. The main properties can now be found and the mathematical aspects as well systematically studied and applied [6, 10, 11, 14, 20]. Some of the more relevant properties of the entropy functions, such as the non decreasing of the von Neumann entropy, and the non increasing of the relative entropy were proved originally in connection with the problem of quantum measurement and of the (irreversible) time evolution of the physical system [1, 8, 9, 17]. Extensions of the concept of the von Neumann entropy have been proposed by

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different authors (e.g. [12, 13, 16]). They have been recently given a unified formulation whose properties have been as well studied [5]. The notion of entropy has been formulated also in the context of Boolean and orthomodular lattices with Bayessian state [7]. Besides the mentioned developments, it seems also of interest a study of the entropy in relation with other basic concepts in quantum mechanics.

In the present paper attention is preliminary focused on the relation between relative entropy and superposition of states. It is shown, as mentioned in [24], that a finite value of the relative entropy between statistical operators implies the existence of a superposition relation between them. In case of relative entropy of tensor product states the superposition relation of the states is preserved on the states of both the components of the product. Similar result holds for the reduced states. In case of reduced states of the same finite dimension a "Schmidt-like" state can be constructed. The relative entropy of this state with respect to the tensor product of the reduced states results greater than the entropy of the single reduced state. Moreover it reaches its minimum in correspondence to the maximum of the reduced state.

The problem of the changement of the entropy under dynamical map is then considered. General results existing on the argument are reported in a unified formulation. It is recalled that a dynamical map does preserve the superposition for statistical operators. However the non decreasing of von Neumann entropy is not ensured by the general definition of dynamical map. To have that property a further (technical) condition must be required. Similarly for the relative entropy [6, 11]. Finally the problem of the existence of nonlinear dynamical maps that do not decrease the entropy is raised. As an answer, a class of dynamical maps that do not decrease the entropy is proposed. These maps are characterized, roughly speaking, by a request of dispersion of information. This is expressed by requiring that a statistical operator remains superposition and "greater" in a suitably specified sense, of its transformed under dynamical map.

2 Definitions and Preliminary Results

It is well known that the generalized probability measure on the standard logic $L(H)$ of the closed subspaces of a complex separable Hilbert space H of $\dim H \geq 3$, are represented, by Geason's theorem [2], by the density (statistical) operators $K(H)$ on H. The density operators are the trace class positive operators on H with trace 1. The probability of the outcome "yes" for a test of the proposition $a \in L(H)$ when the physical system is in the state $\rho \in K(H)$ is then given by the expression $\rho(a) = \mathrm{Tr}\, P^a \rho$, P^a being the orthogonal projection on H whose range is a. The mentioned representation has the advantage of giving to general concepts a formulation in terms of statistical operators. This is the case of the superposition relation for the states of a quantum system as well as for the study of concepts that can be expressed as function of the state of the system. Here we are interested in the entropy and the relative entropy functions.

Definition 1 A state $\rho \in K(H)$ is said to be superposition of the states of $D \subset K(H)$ if anyone of the following equivalent conditions holds:

(i) $b \in L(H), \sigma(b) = 1\ \forall \sigma \in D \Rightarrow \rho(b) = 1$
(ii) $[\rho] \leq \bigvee_{\sigma \in D} [\sigma]$

where $[\rho] (\in L(H))$ denotes the range of ρ as an operator in H. The symbols $\leq, \bigvee$ denote, respectively, the order and the least upper bound relations of the standard logic $L(H)$.

The definition (i) was originally proposed by Varadarajan [18] for a general quantum logic. The equivalence, in case of density operators [15] of the two conditions, follows by considering the spectral decomposition of a density operator e.g. [3, 21, 22]. Since the pure states are represented by one-dimensional projections, it is easily seen that the Definition 1(ii) includes the notion of pure superposition of pure states as well as the statistical mixture of statistical operators. Of course the definition applies also for D containing one only state. (In case of the normal states of a $W^\star$-algebra the condition (ii) is expressed by $\operatorname{supp}\rho \leq \bigvee_{\sigma\in D}\operatorname{supp}\sigma$ [22].)

Definition 2 The quantum entropy $S(\rho)$ relative to a state $\rho \in K(H)$ and the relative quantum entropy $S(\rho|\sigma)$ of the states $\rho,\sigma \in K(H)$ are given respectively by $S(\rho) = -\operatorname{Tr}\rho\log\rho$, and $S(\rho|\sigma) = \operatorname{Tr}\rho(\log\rho - \log\sigma)$.

The definition of the quantum entropy is due to von Neumann [19]. The meaning of the relative entropy follows by reading $S(\rho|\sigma) = -\operatorname{Tr}\rho\log\sigma - S(\rho)$. Review of the properties of the entropy and of the relative entropy can be found in [6, 11, 14, 20]. Here we simply recall that $S(\rho) \geq 0$ for every ρ while the equality holds if and only if ρ is a pure state. Similarly $S(\rho|\sigma) \geq 0$ and the equality holds if and only if $\rho = \sigma$. A systematic study and application of the entropy concepts can be found in [6, 10, 11]. It is of interest the behaviour of the entropy under time evolution and under measurement of the physical system.

Definition 3 A dynamical map B is a map of the statistical operators $K(H)$ into themselves that is affine, that is: $B(\alpha\rho + (1-\alpha)\sigma) = \alpha B\rho + (1-\alpha)B\sigma$ for every $\rho,\sigma \in K(H), \alpha \in [0]$.

A dynamical map is not assumed to be onto nor one-to-one. Examples of dynamical maps are given by the dynamics of a quantum system coupled to its surroundings. The time evolution is in this case represented by a one-parameter family $t \to B_t$ of dynamical maps [4, 8, 9]. Alternatively a dynamical map can represent a projective measurement of the quantum system [8–10].

3 Entropy, Superposition and Tensor Product

An aspect of interest in the definition of the relative entropy is that it essentially takes finite value for states that are in a superposition relation.

Lemma 1 *Let* $\rho,\sigma \in K(H)$. *Then* $-\operatorname{Tr}\rho\log\sigma < +\infty \Rightarrow [\rho] \leq [\sigma]$.

Proof Consider the spectral decomposition $\sigma = \sum_j \sigma_j P^{\phi_j}$ so that it holds $\vee_h[\phi_h] = [\sigma]$. Consider the complete orthonormal system in H given by $\{\tilde{\phi}_l\} = \{\phi_h\} \cup \{\phi'_i\} \cup \{\phi''_k\}$ where $\{\phi_h\} \cup \{\phi'_i\}$ generates $[\rho] \vee [\sigma]$. Then one has

$$-\operatorname{Tr}\rho\log\sigma = -\sum_h \log\sigma_h\langle\phi_h|\rho|\phi_h\rangle - \sum_i\langle\phi'_i|\rho\log\sigma|\phi'_i\rangle$$

If $-\operatorname{Tr}\rho\log\sigma < \infty$ also the second term on the right hand side of the equation must be finite. Hence $\{\phi'_i\}$ must be empty. Therefore $[\rho] \vee [\sigma] = [\sigma]$ or $[\rho] \leq [\sigma]$. □

As a consequence of Lemma 1 one has that if $[\rho] \wedge [\sigma]^\perp \neq \emptyset$ then $-\operatorname{Tr}\rho\log\sigma = \infty$. From the definition of $S(\rho|\sigma) \geq 0$ there also follows that $-\operatorname{Tr}\rho\log\sigma < \infty \Rightarrow S(\rho) < \infty \Rightarrow$

$S(\rho|\sigma) < \infty \Rightarrow [\rho] \leq [\sigma]$. In case $\dim H < \infty$, $S(\rho|\sigma)$ is finite if and only is ρ is superposition of σ. These facts have a counterpart in the tensor product of Hilbert spaces.

Proposition 1 *Let $\rho_1, \sigma_1 \in K(H_1)$, $\rho_2, \sigma_2 \in K(H_2)$ and suppose $-\operatorname{Tr} \rho_1 \otimes \rho_2 \log \sigma_1 \otimes \sigma_2 < +\infty$. Then*

$$S(\rho_1 \otimes \rho_2 | \sigma_1 \otimes \sigma_2) < \infty \quad \Rightarrow \quad [\rho_1] \leq [\sigma_1] \quad \textit{and} \quad [\rho_2] \leq [\sigma_2]$$

Proof The result follows from assumptions and Lemma 1, $[\rho_1 \otimes \rho_2] \leq [\sigma_1 \otimes \sigma_2]$. Hence $[\rho_1 \otimes \rho_2] = [\rho_1] \otimes [\rho_2] \leq [\sigma_1] \otimes [\sigma_2]$ if and only if $[\rho_1] \leq [\rho_2]$ and $[\sigma_1] \leq [\sigma_2]$. The last result is a consequence of the basic properties of tensor product of logics that holds in particular for the Hilbert tensor product of logics [23]. □

Suppose now $\{\phi_{1h}\}, \{\phi_{2k}\}$ be complete orthonormal system in H_1, H_2 respectively. The partial trace operators of a density operator $\sigma \in K(H_1 \otimes H_2)$ are defined by $\sigma_1 = \operatorname{Tr}_2 \sigma = \sum_h \langle \phi_{2h} | \sigma | \phi_{2h} \rangle$ and $\sigma_2 = \operatorname{Tr}_1 \sigma = \sum_h \langle \phi_{1h} | \sigma | \phi_{1h} \rangle$.

Proposition 2 *Let $\rho, \sigma \in K(H_1 \otimes H_2)$ and suppose $-\operatorname{Tr} \rho \log \sigma < +\infty$. Then the reduced state ρ_1 is superposition of σ_1, ρ_2 is superposition of σ_2 and both $S(\rho_1|\sigma_1)$ and $S(\rho_2|\sigma_2)$ take finite values.*

Proof 1 From Proposition 1 ρ is superposition of $\sigma : [\rho] \leq [\sigma]$. Since partial trace preserves superposition [24], one has $[\rho_1] \leq [\sigma_1]$ and $[\rho_2] \leq [\sigma_2]$. □

Proof 2 Partial trace does not increase the relative entropy [14]: $S(\rho_1|\sigma_1) \leq S(\rho|\sigma)$, $S(\rho_2|\sigma_2) \leq S(\rho|\sigma)$. By assumption and by Lemma 1 one has the result. □

If $\dim H_1 \otimes H_1 < \infty$, then $S(\rho|\sigma) < \infty$ if and only if $S(\rho_1|\sigma_1) < \infty$ and $S(\rho_2|\sigma_2) < \infty$. Notice that if the states $\rho_1 \in K(H_1), \rho_2 \in K(H_2)$ have spectral decompositions $\rho_1 = \sum_j^N \alpha_j P^{\phi_{1j}}$, $\rho_2 = \sum_h^N \beta_h P^{\psi_{2h}}$ then one can construct the "Schmidt-like" state [24]

$$\rho(f) = \sum_l^N f_l P^{\phi_{1l}} \otimes P^{\psi_{2l}}$$

for every family of positive numbers such that $\sum_h^N f_h = 1$. Then $\rho(f)$ is a superposition of $\rho_1 \otimes \rho_2$ [24] and the expression $S(\rho(f)|\rho_1 \otimes \rho_2)$ is finite. One obtains

$$S(\rho(f)|\rho_1 \otimes \rho_2) = \sum_1^N f_k (\log f_k - \log \alpha_k \beta_k)$$
$$\geq \sum_1^N (f_k - \alpha_k \beta_k) = 1 - \epsilon$$

with $0 < \epsilon = \sum_1^N \alpha_k \beta_k < 1$ (the inequality $x \log x - x \log y \geq x - y$ has been used). Moreover one can check that if $\alpha_k = \beta_k = f_k = 1/N \forall k$ then one has $S(\rho(f)|\rho_1 \otimes \rho_2) = \log N$. By using the Lagrange multipliers and calculating the involved Hessian, one can also show that $\log N$ is indeed the minimum value of $S(\rho(f)|\rho_1 \otimes \rho_2)$ for fixed N. By recalling that $S(\rho_1)$ takes, as maximum value, the value $\log N$, one has the result

$$S(\rho(f)|\rho_1 \otimes \rho_2) \geq S(\rho_1), S(\rho_2)$$

that holds in general for arbitrary but fixed N.

4 Superposition, Entropy and Dynamical Maps

The problem is how the superposition relation and the entropies change under dynamical map and quantum measurement. For what concerns an isolated physical system it is plausible to assume the dynamics to be of the form $\rho \to B_t\rho = U_t\rho U_t^\dagger$, U_t a one parameter group of unitary operators e.g. [22]. Then B_t is a dynamical map for every t, and, as it has to be, $[\rho] = [B_t\rho]$, $S(B_t\rho) = S(\rho)$, $S(B_t\rho|B_t\sigma) = S(\rho|\sigma)$, $\rho, \sigma \in K(H)$ and for every time t. In general the time evolution and the quantum measurement of a physical system can be characterized by a one parameter family of (or by a single) dynamical map. A first general result is

Theorem 1 ([22]) *A dynamical map B of the physical system preserves superposition*, *that is*: $\rho, \sigma \in K(H), [\rho] \leq [\sigma] \Rightarrow [B\rho] \leq [B\sigma]$.

The linear extension $\tilde{B}$ of a dynamical map B to the trace class operators $T(H)$ on H is a positive trace preserving linear map of $T(H)$ into itself. Its dual $\tilde{B}^\star$, defined by $\mathrm{Tr}(\tilde{B}^\star a)\rho = \mathrm{Tr}\, a(\tilde{B}\rho)$, is a positive map of the bounded operators $B(H)$ into themselves such that $\tilde{B}^\star\mathbf{1} = \mathbf{1}$. According to [6] one has then

$$-\tilde{B}^\star(A \log A) \leq -(\tilde{B}^\star A)\log(\tilde{B}^\star A)$$

for any positive operator $A \in B(H)$. It is clear that the last result does not a priori ensure the non decreasing of $S(\rho)$ under a dynamical map. In fact one has to add a condition:

Theorem 2 ([6]) *Let B a dynamical map such that $\tilde{B}^\star$ preserves the trace on the statistical operators*: $\mathrm{Tr}\,\tilde{B}^\star\rho = \mathrm{Tr}\,\rho$. *Then* $S(B\rho) \geq S(\rho)$.

Proof This is a detail of what done in [6]. By assumptions and the linearity of $\tilde{B}^\star$ one has $\mathrm{Tr}\,\tilde{B}^\star\rho = \mathrm{Tr}\,\rho, \forall\rho \in T(H)$. Moreover $\mathrm{Tr}\,\tilde{B}^\star\rho = \mathrm{Tr}\,\tilde{B}\rho$. Therefore $\tilde{B}^\star\rho = \tilde{B}\rho$. By setting $A = \rho \in K(H)$ in the inequality reported above and taking the trace one has $S(B\rho) \geq S(\rho)$. □

A sufficient condition for a dynamical map B not to decrease the entropy is alternatively given by $\mathrm{Tr}\, a \geq \mathrm{Tr}(B^\star a)$ for every positive operator $a \in T(H)$ [1].

For what concerns the monotonicity of the relative entropy, it well known that the complete positivity of B implies: $S(B\rho|B\sigma) \leq S(\rho|\sigma) \forall\rho, \sigma \in K(H)$ [8–11, 17]. More generally, the monotonicity is ensured for B linear mapping on the normal states of a von Neumann algebra [6].

One could now ask if it is possible to obtain the non decreasing of the von Neumann entropy under dynamical map not necessarily affine. To have an answer let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$, $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots$ be two decreasing sequence of positive numbers with

$\sum_i \alpha_i = \sum_i \beta_i = 1$ such that $\sum_1^N \alpha_i \leq \sum_1^N \beta_i$, $N = 1, 2, 3, \ldots$. We denote this fact by writing $(\beta_i) \succ (\alpha_i)$ [20]. The concept is used for introducing an order in $K(H)$: if the α_i's and the β_i's represent the eigenvalues of the statistical operators ρ, σ respectively one can indeed write $\rho \prec \sigma$ (e.g. [5]). It is possible to extend that order also to vectors and density operators. Indeed let $\psi \in H$, $\|\psi\| = 1$, $\rho \in K(H)$ with spectral decomposition $\rho = \sum_i \alpha_i P^{\phi_i}$, $\|\phi_i\| = 1$. If $\psi = \sum_h c_h \phi_h$ we write $(\psi) \succ \rho$ if it happens $(|c_k|^2) \succ (\alpha_i)$. When this is the case it holds $\sum_i \alpha_i \log \alpha_i \leq \sum_k |c_k|^2 \log \alpha_k \leq \sum_k |c_k|^2 \log |c_k|^2$ [20]. It is now possible to give a positive answer to the above question.

Proposition 3 *Let B a not necessarily affine dynamical map and consider, for $\rho \in K(H)$, the spectral decompositions $\rho = \sum_i \alpha_i P^{\phi_i}$, $B\rho = \sum_k \beta_k P^{\psi_k}$ ($\|\psi_i\| = \|\phi_i\| = 1$). Suppose the following conditions hold*: (i) $[\rho] \leq [B\rho]$ *and* (ii) $(\phi_h) \succ B\rho\ \forall \rho \in K(H), h = 1, 2, \ldots$. *Then it holds $S(B\rho) \geq S(\rho)$.*

Proof From assumption (i) one can write $\phi_j = \sum_l c_l^j \psi_l \forall j$. By using the expression $P^{\phi_j} = |\phi_j\rangle\langle\phi_j| \forall j$, one obtains:

$$\begin{aligned} -\operatorname{Tr} \rho \log(B\rho) &= -\sum_{kj} \log \beta_k \langle \psi_k | P^{\phi_j} | \psi_k \rangle \alpha_j \\ &= -\sum_{kj} \alpha_j |c_k^j|^2 \log \beta_k \\ &\leq -\sum_{kj} \alpha_j \beta_k \log \beta_k = -\sum_k \beta_k \log \beta_k \\ &\leq -\operatorname{Tr}(B\rho) \log(B\rho) \end{aligned}$$

where the condition (ii) has been used. Therefore

$$\begin{aligned} S(\rho|B\rho) &= -S(\rho) - \operatorname{Tr} \rho \log(B\rho) \\ &\leq -S(\rho) - \operatorname{Tr}(B\rho) \log(B\rho) = -S(\rho) + S(B\rho). \end{aligned}$$

By the positivity of the relative entropy one has the result. □

The condition (i) of the proposition means that the spectral decomposition of $B\rho$ contains a number of one dimensional eigenprojection not smaller than that of ρ. Together with the condition (ii) this is a possible formalisation of the idea of " dispersion of information" under dynamical map.

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